Appendix of Particle-Based Anisotropic Surface Meshing

1 Relation with Fattal's Formulation

In this section, we show the fundamental relationship between our particle-based formulation and Fattal's kernel-based formulation [1]. In particular, for the uniform isotropic case, we show in Sec. 1.1 that our particle-based formulation is equivalent to Fattal's kernel-based formulation. For the non-uniform isotropic case, we show in Sec. 1.2 how the difference of these two formulation leads to the result shown in Fig. 2.

1.1 Uniform Isotropic Case

We start from Fattal's kernel-based formulation. Given n samples with their positions $\mathbf{X} = \{\mathbf{x}_i | i = 1 \dots n\}$ in the domain $\Omega \subset \mathbb{R}^m$, we place the Gaussian kernel centered around each sample position $\mathbf{x}_i$:

$$G_i(\mathbf{x}) = \frac{1}{(\sqrt{2\pi}\sigma)^2} e^{-\frac{\|\mathbf{x}-\mathbf{x}_i\|^2}{2\sigma^2}}, \tag{1.1}$$

where σ is the standard deviation of Gaussian kernel, and $\mathbf{x} \in \Omega$. We call σ the "kernel width", and assume that all kernels have the same fixed width. Here the normalization factor $\frac{1}{(\sqrt{2\pi}\sigma)^2}$ ensures that the kernel's integral to be fixed. Given any target function $C(\mathbf{x})$, Fattal [1] suggests to use the summation of kernels to approximate $C(\mathbf{x})$, i.e., to minimize the following energy:

$$E(\mathbf{X}) = \int_\Omega |C(\mathbf{x}) - \sum_{i=1}^{n} G_i(\mathbf{x})|^2 \, \mathrm{d}s. \tag{1.2}$$

If we set the target function to be a unit constant $C(\mathbf{x}) = 1$, then minimizing $E(\mathbf{X})$ w.r.t. the sample positions $\mathbf{X}$ will achieve a hexagonal distribution of samples in 2D space [1].

The minimizer is defined by: $\mathbf{X}^* =$

$$\arg\min_{\mathbf{X}} \left\{ \underbrace{\int_\Omega \mathrm{d}s}_{\text{constant}} \underbrace{-2 \int_\Omega (\sum_{i=1}^{n} G_i(\mathbf{x})) \mathrm{d}s}_{\text{constant}} + \int_\Omega (\sum_{i=1}^{n} G_i(\mathbf{x}))^2 \mathrm{d}s \right\}.$$

Let us suppose the domain Ω is finite, either open or closed. If Ω is open with boundaries, we can constrain the samples on a slightly smaller domain $\Omega' \subset \Omega$ so that the finite supports of these kernels are all in Ω. Under this assumption, we can see that only the third term $\int_\Omega (\sum_{i=1}^{n} G_i(\mathbf{x}))^2 \mathrm{d}s$ depends on the sample location $\mathbf{X}$. The first term is obviously constant. The second term is also constant because the Gaussian kernels are normalized. Thus, the above minimizer can be simplified as:

$$\begin{aligned}
&\arg\min_{\mathbf{X}} \int_\Omega (\sum_{i=1}^{n} G_i(\mathbf{x}))^2 \mathrm{d}s \\
&= \arg\min_{\mathbf{X}} \sum_{i=1}^{n} \sum_{j=1, j\neq i}^{n} \frac{1}{(2\pi\sigma^2)^2} e^{-\frac{\|\mathbf{x}_i - \mathbf{x}_j\|^2}{4\sigma^2}} \int_\Omega e^{-\frac{\|\mathbf{x} - \frac{\mathbf{x}_i + \mathbf{x}_j}{2}\|^2}{\sigma^2}} \mathrm{d}s \\
&= \arg\min_{\mathbf{X}} \sum_{i=1}^{n} \sum_{j=1, j\neq i}^{n} \frac{1}{(2\sigma\sqrt{\pi})^2} e^{-\frac{\|\mathbf{x}_i - \mathbf{x}_j\|^2}{4\sigma^2}} \\
&= \arg\min_{\mathbf{X}} \sum_{i=1}^{n} \sum_{j=1, j\neq i}^{n} \frac{1}{(2\sigma\sqrt{\pi})^2} E^{ij}(\mathbf{X}) \\
&= \arg\min_{\mathbf{X}} \sum_{i=1}^{n} \sum_{j=1, j\neq i}^{n} E^{ij}(\mathbf{X}).
\end{aligned} \tag{1.3}$$

Here E^{ij} is the inter-particle energy as we defined in our paper. From this derivation, we can see that Fattal's kernel-based formulation [1] is fundamentally equivalent to our particle-based formulation for uniform isotropic case, where only the mutual effect between particles i and j ($i \neq j$) needs to be considered. Such inter-particle repulsion model avoids the numerical integration of the kernel-based functional approximation, making it practical to handle anisotropic meshing with large stretching ratios.

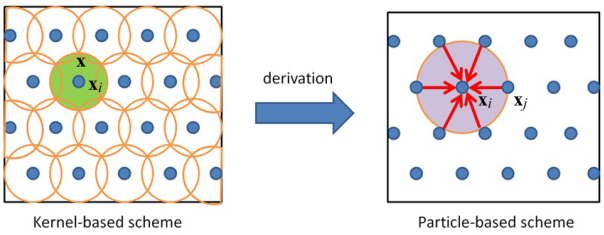

Figure 1: *Equivalence between Fattal's kernel-based scheme and our particle-based scheme, for the uniform isotropic case.*

1.2 Non-Uniform Isotropic Case

For non-uniform isotropic case with background density $\rho(\mathbf{x})$ defined on a 2D domain, the metric tensor is essentially $\mathbf{M}(\mathbf{x}) = \rho(\mathbf{x}) \cdot \mathbf{I}$, where $\mathbf{I}$ is the identity matrix. Let us denote the particle positions on the corresponding surface $\overline{\Omega}$ in the high-dimensional embedding space by $\overline{\mathbf{X}} = \{\overline{\mathbf{x}}_i \mid \overline{\mathbf{x}}_i = \phi(\mathbf{x}_i), i = 1 \dots n\}$. In our paper we define the following inter-particle energy for particles i and j in the embedding space:

$$\overline{E}^{ij} = e^{-\frac{\|\overline{\mathbf{x}}_i - \overline{\mathbf{x}}_j\|^2}{4\sigma^2}}. \tag{1.4}$$

Using the derivation introduced in the above section, we can see that our particle-based energy formulation, in the embedding space, is fundamentally equivalent to the optimization of the following

kernel-based energy function:

$$\overline{E}(\overline{\mathbf{X}}) = \int_{\overline{\Omega}} |1 - \sum_{i=1}^{n} G_i(\overline{\mathbf{x}})|^2 \, \mathrm{d}\overline{s}, \tag{1.5}$$

where $G_i(\overline{\mathbf{x}})$ is the normalized Gaussian kernel in $\overline{\Omega}$, and is defined as:

$$G_i(\overline{\mathbf{x}}) = \frac{1}{(\sqrt{2\pi}\sigma)^2} e^{-\frac{\|\overline{\mathbf{x}} - \overline{\mathbf{x}}_i\|^2}{2\sigma^2}}. \tag{1.6}$$

Fattal's kernel-density energy function (Eq. (3) of [1]) in 2D domain can be summarized as:

$$E'(\mathbf{X}) = \int_{\Omega} |\rho(\mathbf{x}) - \sum_{i=1}^{n} G'_i(\mathbf{x})|^2 \, \mathrm{d}s, \tag{1.7}$$

where $G'_i(\mathbf{x})$ is a normalized Gaussian kernel centered around $\mathbf{x}_i$ in Ω:

$$G'_i(\mathbf{x}) = \frac{\rho(\mathbf{x})}{(\sqrt{2\pi}\sigma)^2} e^{-\frac{\rho(\mathbf{x})\|\mathbf{x} - \mathbf{x}_i\|^2}{2\sigma^2}}. \tag{1.8}$$

If we want to map Fattal's kernel-density energy to $\overline{\Omega}$ in the high-dimensional embedding space, we can introduce $\phi : \Omega \to \overline{\Omega}$, and exploit the following relationship:

$$\begin{aligned} \|\overline{\mathbf{x}} - \overline{\mathbf{x}}_i\|^2 &= < \overline{\mathbf{x}} - \overline{\mathbf{x}}_i, \overline{\mathbf{x}} - \overline{\mathbf{x}}_i > \\ &= (\mathbf{x} - \mathbf{x}_i)^T \mathbf{J}(\mathbf{x})^T \mathbf{J}(\mathbf{x})(\mathbf{x} - \mathbf{x}_i) \\ &= (\mathbf{x} - \mathbf{x}_i)^T \mathbf{M}(\mathbf{x})(\mathbf{x} - \mathbf{x}_i) \\ &= \rho(\mathbf{x})\|\mathbf{x} - \mathbf{x}_i\|^2, \end{aligned} \tag{1.9}$$

where $\mathbf{J}(\mathbf{x})$ denotes the Jacobian matrix of the embedding function ϕ evaluated at $\mathbf{x}$. Thus the Gaussian kernel of Eq. (1.8) can be rewritten as:

$$G'_i(\overline{\mathbf{x}}) = \frac{\rho(\phi^{-1}(\overline{\mathbf{x}}))}{(\sqrt{2\pi}\sigma)^2} e^{-\frac{\|\overline{\mathbf{x}} - \overline{\mathbf{x}}_i\|^2}{2\sigma^2}}.$$

Considering the infinitesimal area of $\overline{\Omega}$: $\mathrm{d}\overline{s} = \rho(\mathbf{x})\mathrm{d}s$, we can rewrite Fattal's kernel-density energy of Eq. (1.7) in $\overline{\Omega}$ as:

$$\overline{E}(\overline{\mathbf{X}}) = \int_{\overline{\Omega}} \rho(\phi^{-1}(\overline{\mathbf{x}}))|1 - \sum_{i=1}^{n} G_i(\overline{\mathbf{x}})|^2 \, \mathrm{d}\overline{s}, \tag{1.10}$$

where $G_i(\overline{\mathbf{x}}) = \frac{1}{(\sqrt{2\pi}\sigma)^2} e^{-\frac{\|\overline{\mathbf{x}} - \overline{\mathbf{x}}_i\|^2}{2\sigma^2}}$ is the normalized Gaussian kernel on $\overline{\Omega}$, and $\rho(\phi^{-1}(\overline{\mathbf{x}})) = \rho(\mathbf{x})$.

By comparing Eq. (1.5) with Eq. (1.10), which are both formulated on the "isotropic" $\overline{\Omega}$ in the high-dimensional embedding space, we can see that our energy of Eq. (1.5) is a formulation of least squares, while Fattal's energy of Eq. (1.10) is the weighted least squares (weighted by $\rho(\phi^{-1}(\overline{\mathbf{x}}))$).

As can be seen, our approach leads to a formulation that is slightly different from Fattal et al.'s approach, when handling non-uniform isotropic case with given background density. Our following result shows that Fattal's kernel-density energy function cannot produce desirable result.

We use L-BFGS optimizer to directly minimize Fattal's energy in Eq. (1.7), and compare with our particle-based optimization result, both running in a 2D square domain with certain density functions. Fig. 2 shows such comparative results with 500 sample points. We can see that by using Fattal's energy function, all the samples are attracted to the high density area, which does not match the input density field. In contrast, our particle-based energy optimization can generate the correct sampling result conforming to the input density.

Note that Fattal's method does not handle anisotropic case.

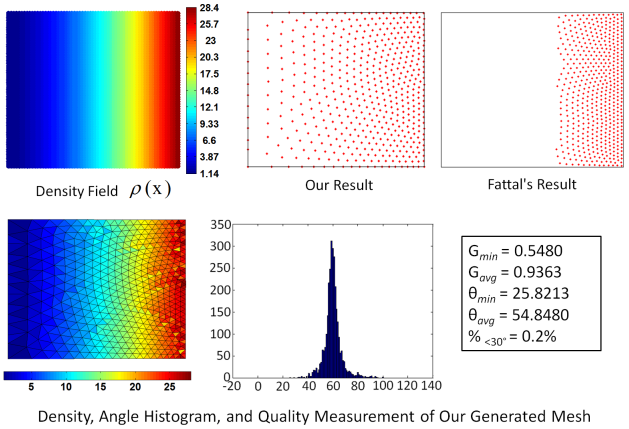

Figure 2: *The sampling results with 500 particles by optimizing our particle-based energy function and Fattal's kernel-density energy function.*

2 A Simple 1D Example to Compare Forces

In this section, we show a simple 1D example to illustrate the difference between our force (Eq. (21) of the paper) and the other two alternative forces (Eq. (24) and Eq. (25) of the paper), as illustrated in the following Fig. 3.

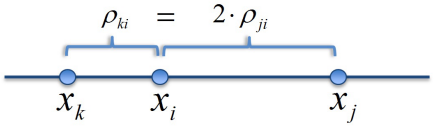

Figure 3: *A simple 1D example with particles k, i, and j.*

Suppose particles j and k are two neighbors of particle i. They are placed in the 1D domain with desired density $\rho_{ki} = 2\rho_{ji}$. Then we will expect to have $\|\mathbf{x}_j - \mathbf{x}_i\| = 2\|\mathbf{x}_k - \mathbf{x}_i\|$ using a correct force function. Since it is 1D case, according to Eq. (3) of our paper, we know the metric $\mathbf{M} = \rho^2$, and $\mathbf{Q} = \rho$. If we use our force function in Eq. (21) of the paper, then:

$$\begin{aligned} \widetilde{\mathbf{F}}_{ki} &= \frac{\rho_{ki}(\mathbf{x}_k - \mathbf{x}_i)}{2\sigma^2} e^{-\frac{\rho_{ki}^2\|\mathbf{x}_k - \mathbf{x}_i\|^2}{4\sigma^2}}, \\ \widetilde{\mathbf{F}}_{ji} &= \frac{\rho_{ji}(\mathbf{x}_j - \mathbf{x}_i)}{2\sigma^2} e^{-\frac{\rho_{ji}^2\|\mathbf{x}_j - \mathbf{x}_i\|^2}{4\sigma^2}}. \end{aligned} \tag{2.1}$$

If we sum up the two forces applied on particle i: $\widetilde{\mathbf{F}}_i = \widetilde{\mathbf{F}}_{ki} + \widetilde{\mathbf{F}}_{ji}$, we can see that:

$$\widetilde{\mathbf{F}}_i = 0 \quad \Leftrightarrow \quad \|\mathbf{x}_j - \mathbf{x}_i\| = 2\|\mathbf{x}_k - \mathbf{x}_i\|, \tag{2.2}$$

which means that the force equilibrium is exactly located at the desired particle configuration. It is easy to see that using the other two alternative forces in Eq. (24) and Eq. (25) of the paper:

$$\widehat{\mathbf{F}}_i = 0 \quad \Leftrightarrow \quad \|\mathbf{x}_j - \mathbf{x}_i\| = 2\|\mathbf{x}_k - \mathbf{x}_i\|. \tag{2.3}$$

References

[1] R. Fattal, *Blue-noise point sampling using kernel density model*, ACM Transactions on Graphics **30** (2011), no. 4, 48:1–48:12.